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Upper Quantum Lyapunov Exponent and Anosov Relations for Quantum Systems Driven by a Classical Flow

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We generalize the definition of quantum Anosov properties and the related Lyapunov exponents to the case of quantum systems driven by a classical flow, i.e. skew-product systems. We show that the skew Anosov properties can be interpreted as regular Anosov properties in an enlarged Hilbert space, in the framework of a generalized Floquet theory. This extension allows us to describe the hyperbolicity properties of almost-periodic quantum parametric oscillators and we show that their upper Lyapunov exponents are positive and equal to the Lyapunov exponent of the corresponding classical parametric oscillators. As second example, we show that the configurational quantum cat system satisfies quantum Anosov properties.

KEY WORDS: quantum dynamics, Lyapunov exponents, Anosov systems, parametric oscillators, quantum chaos, Arnold's cat map

1. INTRODUCTION

Anosov properties and Lyapunov exponents are well-established characterization of classical dynamics and it is natural to search for similar concepts applicable to quantum dynamics. Several definitions have been given in the literature (see Refs. 1, 6, 13–18, 20, 21, 24 and the references therein).

Majewski and Kuna[(14)] defined a quantum Lyapunov exponent for N-level quantum systems. Later,[3] Emch, Narnhofer, Sewell and Thirring[(1,20,24)] proposed

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[3] Erratum to Ref. 6: The chronology of the Refs. 1 and 14 as described in Ref. 6 by two of the present authors is erroneous. To our knowledge the works of Refs. 1 and 14 were developed independently, while[(14)] was published before.[(1)]

an axiomatic framework which allows one to define an Anosov property for quantum mechanical systems. However, the resulting definition of a quantum Lyapunov exponent is limited since it only applies to systems with a globally constant hyperbolicity property.

In Ref. 6, the upper Lyapunov exponent for quantum systems in the Heisenberg representation has been defined, close in spirit to definitions given in Refs. 1, 14. Its usefulness has been illustrated with the example of the parametric quantum oscillator with periodic time dependence. Moreover, it was shown that whenever its upper Lyapunov exponent is positive, the system satisfies the discrete quantum Anosov relations defined by Emch, Narnhofer, Sewell and Thirring.[1,20,24]

In this paper we extend the study to systems described by a Hamiltonian operator of the form $H(\varphi^t(\theta))$ (with φ^t a flow on a space $\mathcal{M}$), which will be referred to as *quantum skew-product system*. We generalize the definition of Anosov relations so that it applies to this type of system. As in the case of Floquet theory,[2–5,27] it is possible to make quantum skew-product systems autonomous by embedding the dynamics in a larger Hilbert space. The Anosov relations for quantum skew-product systems correspond to the Anosov relations of the associated system in this enlarged Hilbert space. We consider the parametric oscillator as an example. We show that the quantum parametric oscillator verifies the Anosov relations for quantum skew-product systems if its upper Lyapunov exponent is positive and the corresponding classical dynamics is reducible (see Definition 5 or Ref. 12). Thus the quantum parametric oscillator discussed in Ref. 6 is an Anosov quantum skew-product system. As a second example we consider the configurational quantum cat system,[25,26] with periodic boundary conditions, which amounts to a system with compact configuration space.

This paper is organized in the following way: In Sec. 2 we recall the definition of the upper Lyapunov exponent and of the Anosov properties for a quantum system describing the motion of a particle. In Sec. 3, we present the formalism of quantum skew-product systems and the enlarged Hilbert space which allows one to turn the system into an autonomous one. We propose a definition of the quantum Anosov properties for quantum skew-product systems in Sec. 4, and illustrate it by treating the example of the almost-periodic quantum parametric oscillator in Sec. 5. Finally, the configurational cat map is studied in Sec. 6 after adapting the definition of the Anosov property to systems with a toroidal configuration space.

2. UPPER LYAPUNOV EXPONENTS AND QUANTUM ANOSOV RELATIONS

A quantum mechanical particle on the real line is described by coordinate and momentum operators $\hat{x}$ and $\hat{p}$ which satisfy the Heisenberg commutation relation (we choose the units such that $\hbar = 1$):

$$[\hat{x}, \hat{p}] = i.$$

We consider the algebra $\mathcal{W}$ of finite linear combinations of Weyl operators:

$$W(\beta,\gamma)=\exp[i(\beta\hat{x}+\gamma\hat{p})],\quad \forall\beta,\gamma\in\mathbb{R}.$$

These operators satisfy the Weyl form of the commutation relations:

$$W(\beta,\gamma)^{\dagger}=W(-\beta,-\gamma),$$
$$W(\beta,\gamma)\,W(\beta',\gamma')=e^{-\frac{i}{2}(\beta\gamma'-\gamma\beta')}W(\beta+\beta',\gamma+\gamma').$$

More abstractly, if the phase space is a real symplectic space V with symplectic form σ, the algebra $\mathcal{W}$ over (V,σ) is defined as the algebra of finite linear combinations of the elements $\{W(\underline{\alpha})\,|\,\underline{\alpha}\in V\}$ such that

$$W(\underline{\alpha})^{\dagger}=W(-\underline{\alpha}), \tag{1}$$

$$W(\underline{\alpha})\,W(\underline{\alpha}')=e^{-\frac{i}{2}\sigma(\underline{\alpha},\underline{\alpha}')}\,W(\underline{\alpha}+\underline{\alpha}')\;=\;e^{-i\,\sigma(\underline{\alpha},\underline{\alpha}')}\,W(\underline{\alpha}')\,W(\underline{\alpha}). \tag{2}$$

In this paper we consider only phase spaces V of finite dimension $2n$, with the usual symplectic form

$$\sigma(\underline{\alpha},\underline{\alpha}')=\alpha_x^{\mathsf{T}}\alpha_p'\;-\;\alpha_p^{\mathsf{T}}\alpha_x'\qquad \forall\underline{\alpha}=\begin{pmatrix}\alpha_x\\ \alpha_p\end{pmatrix},\quad \underline{\alpha}'\in\mathbb{R}^{2n},$$

where α_x^{T} denotes the transposed of α_x. Hence, the Weyl operators can be written as:

$$W(\underline{\alpha})=\exp\left[i\left(\alpha_x^{\mathsf{T}}\hat{x}+\alpha_p^{\mathsf{T}}\hat{p}\right)\right],\qquad \underline{\alpha}\in\mathbb{R}^{2n}.$$

In order to define the quantum Lyapunov exponent, we consider derivations on the algebra $\mathcal{W}$. We denote by $\delta_{\underline{\alpha}}$ the derivation defined as the generator of the automorphism $M\mapsto W(t\,\underline{\alpha})\,MW(-t\underline{\alpha})$ for all $A\in\mathcal{W}$. Therefore we have

$$\delta_{\underline{\alpha}}(M)\;\equiv\;[L_{\underline{\alpha}},\;M],\qquad \forall\,M\in\mathcal{W},$$

where [,] is the commutator and

$$L_{\underline{\alpha}}=\alpha_x^{\mathsf{T}}\hat{x}\;+\;\alpha_p^{\mathsf{T}}\hat{p},\qquad \underline{\alpha}\in\mathbb{R}^{2n}.$$

In particular, we can check that

$$[L_{\underline{\alpha}},\;W(\underline{\alpha}')]=-\,\sigma(\underline{\alpha},\underline{\alpha}')\,W(\underline{\alpha}'),\qquad \forall\,\underline{\alpha},\underline{\alpha}'\in V. \tag{3}$$

We assume that the dynamics defines an automorphism of $\mathcal{W}$:

$$U^{\dagger}(t,t_0)\;MU(t,t_0)\;\equiv\;M(t,t_0)\in\mathcal{W},\quad \forall\,M\in\mathcal{W},\,\forall\,t,t_0\in\mathbb{R},$$

where $U(t,t_0)$ denotes the unitary propagator with initial time t_0.

Definition 1. (cf.[6]). *The upper quantum Lyapunov exponent is defined as*

$$\bar{\lambda}=\sup_{\underline{\alpha}\in V}\,\bar{\lambda}_{\underline{\alpha}}$$

where

$$\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, M, t_0) := \limsup_{t\to\infty} \frac{1}{t} \ln \| [L_{\underline{\alpha}},\ M(t, t_0)] \|,$$

and the norm is chosen as $\|M\| = \sup_{\psi\in\mathcal{H}} \|M\psi\|/\|\psi\|$.

Remark 1. *According to (3), the derivation* $\delta_{\underline{\alpha}}$ *is well defined on* $\mathcal{W}$. *Hence, the norm of the commutator in the definition of the quantum Lyapunov exponent is finite even the operator* $L_{\underline{\alpha}}$ *is unbounded.*

Since the time evolution is unitary, the exponent $\bar{\lambda}_{\underline{\alpha}}$ can also be expressed as

$$\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, M, t_0) = \limsup_{t\to\infty} \frac{1}{t} \ln \| [L_{\underline{\alpha}}(t_0, t),\ M] \|, \tag{4}$$

with

$$L_{\underline{\alpha}}(t_0, t) := U^{\dagger}(t_0, t) L_{\underline{\alpha}} U(t_0, t).$$

In the examples of Sec. 5, we will use the following

Lemma 1. *If the quantum Lyapunov exponent of Weyl operators* $W(\underline{\beta})$ *is independent of the choice of* $\underline{\beta}$:

$$\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, W(\underline{\beta}), t_0) = \lambda,$$

then for any observable $M \in \mathcal{W}$:

$$\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, M, t_0) = \lambda.$$

Proof: By definition

$$\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, W(\underline{\beta}), t_0) = \limsup_{t\to\infty} \frac{1}{t} \ln \| [L_{\underline{\alpha}}(t_0, t),\ W(\underline{\beta})] \| = \lambda,$$

therefore if $M = \sum_{j=1}^{N} w_j\ W(\underline{\beta}_j)$, the following function

$$c(t, t_0) = \max_{j=1,\ldots,N} \{\| [L_{\underline{\alpha}}(t_0, t),\ W(\underline{\beta}_j)] \|,\ 1\}$$

satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \ln c(t, t_0) = \lambda.$$

Hence

$$\begin{aligned}\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, M, t_0) &= \limsup_{t\to\infty} \frac{1}{t} \ln \| [L_{\underline{\alpha}}(t_0, t),\ M] \| \\ &= \limsup_{t\to\infty} \frac{1}{t} \ln \left\| \sum_{j=1}^{N} w_j [L_{\underline{\alpha}}(t_0, t), W(\underline{\beta}_j)] \right\| \\ &= \limsup_{t\to\infty} \frac{1}{t} \ln c(t, t_0) + \frac{1}{t} \ln \left\| \sum_{j=1}^{N} \frac{w_j}{c(t, t_0)} [L_{\underline{\alpha}}(t_0, t), W(\underline{\beta}_j)] \right\| \\ &= \lambda,\end{aligned}$$

because the term

$$\begin{aligned}\left\| \sum_{j=1}^{N} \frac{w_j}{c(t, t_0)} [L_{\underline{\alpha}}(t_0, t),\ W(\underline{\beta}_j)] \right\| &\leq \sum_{j=1}^{N} \frac{|w_j|}{c(t, t_0)} \ \| [L_{\underline{\alpha}}(t_0, t),\ W(\underline{\beta}_j)] \| \\ &\leq \sum_{j=1}^{N} |w_j|\end{aligned}$$

is bounded. □

Definition 2. *A system satisfies the quantum Anosov relations,*[1,20,24] *if there are* $2n$ *directions* $\underline{\alpha}_1, \ldots, \underline{\alpha}_{2n} \in V$ *such that the corresponding derivations satisfy for all* $t, t_0 \in \mathbb{R}$

$$U(t, t_0)\, L_{\underline{\alpha}_j} U^{\dagger}(t, t_0) = e^{\lambda_j (t-t_0)} L_{\underline{\alpha}_j}, \tag{5}$$

where λ_i *are* $2n$ *complex numbers such that*

$$\mathrm{Re}(\lambda_1) \leq \cdots \leq \mathrm{Re}(\lambda_n) < 0 < \mathrm{Re}(\lambda_{n+1}) \leq \cdots \leq \mathrm{Re}(\lambda_{2n}).$$

Remark 2. *We have extended the definition of Ref. 1 by allowing the numbers* λ_j *to have an imaginary part. Moreover, we do not require that a state invariant under the actions of* U *and* $L_{\underline{\alpha}_j}$ *exist.*

Remark 3. *The property (5) can be written equivalently as*

$$U(t, t_0)\, e^{i\, s\, L_{\underline{\alpha}_j}} = e^{i\, s\, e^{\lambda_j (t-t_0)} L_{\underline{\alpha}_j}}\, U(t, t_0) \qquad \forall s \in \mathbb{R}.$$

This is to be interpreted as a relation between operators acting on the algebra of observables $\mathcal{A}$.

This property yields a representation of the Anosov group[24] *as a group of endomorphisms on the algebra of observables $\mathcal{A}$, defined as*

$$\tau(t, t_0) : M \mapsto U(t, t_0)\, M\, U^{\dagger}(t, t_0)$$

and

$$\sigma_i(s) : M \mapsto e^{i\, s\, L_{\underline{\alpha}_j}}\, M e^{-i\, s\, L_{\underline{\alpha}_j}}.$$

The Anosov property can be expressed as

$$\tau(t, t_0)\, \sigma_j(s) = \sigma_j\left(s e^{\lambda_j (t-t_0)}\right) \tau(t, t_0).$$

3. QUANTUM SKEW-PRODUCT SYSTEMS AND ENLARGED HILBERT SPACE

A quantum skew-product system is described by the following Schrödinger equation with a non autonomous Hamiltonian in a Hilbert space $\mathcal{H}$:

$$i \frac{d}{dt} \phi(t) = H(\varphi^t(\theta))\, \phi(t), \tag{6}$$

where φ^t is a continuous flow on a compact metric space $\mathcal{M}$ while $H(\theta)$ is a self-adjoint operator depending on the parameter $\theta \in \mathcal{M}$ such that the evolution operator $U(t, t_0; \theta)$ exists and is strongly continuous with respect to $\theta \in \mathcal{M}$. This form of Hamiltonian operator includes periodic, quasi-periodic and almost-periodic time dependence according to whether $\mathcal{M}$ is a circle, a torus or the hull of an almost-periodic function.

Any solution of (6) can be written as

$$\phi(t; \theta) = U(t, t_0; \theta)\, \phi(t_0; \theta),$$

with the operator $U(t, t_0; \theta)$ satisfying

$$i \frac{\partial}{\partial t} U(t, t_0; \theta) = H(\varphi^t(\theta))\, U(t, t_0; \theta)$$

and $U(t_0, t_0; \theta) = \mathbb{1}_{\mathcal{H}}$.

The uniqueness of solutions of (6) allows us to deduce the relations

$$U(t, t_1; \theta)\, U(t_1, t_0; \theta) = U(t, t_0; \theta),$$
$$U(t + \tau, t_0 + \tau; \theta) = U(t, t_0; \varphi^{\tau}(\theta)),$$

for all $t, t_0, t_1, \tau \in \mathbb{R}$ and all $\theta \in \mathcal{M}$.
Let μ be an invariant probability measure on $\mathcal{M}$. The family of Koopman operators $(\mathcal{T}^t)_{t \in \mathbb{R}}$, defined by

$$(\mathcal{T}^t \psi)(\theta) = \psi(\varphi^t(\theta)) \quad \text{for all} \quad \psi \in \mathbb{L}^2(\mathcal{M}, d\mu),$$

is a strongly continuous one-parameter unitary group of operators (see Ref. 22). According to Stone's theorem, there exists a self-adjoint operator G which is an infinitesimal generator of $\mathcal{T}^t$:

$$\mathcal{T}^t = e^{itG} \quad \text{for all} \quad t \in \mathbb{R}.$$

The separable Hilbert space $\mathcal{K} = \mathbb{L}^2(\mathcal{M}, d\mu; \mathcal{H}) = \mathbb{L}^2(\mathcal{M}, d\mu) \otimes \mathcal{H}$ will be called *the enlarged space* of $\mathcal{H}$. The family of operators $U(t, t_0; \theta) \in \mathcal{H}$ depending on the parameter $\theta \in \mathcal{M}$ defines a unitary operator acting in $\mathcal{K}$ which maps a function $\theta \mapsto \psi(\theta) \in \mathcal{H}$ of $\mathcal{K}$ to the function $\theta \mapsto U(t, t_0; \theta)\psi(\theta) \in \mathcal{H}$. To avoid a complicated notation, we also denote this operator by $U(t, t_0; \theta)$. Moreover, we omit the identity factor of $\mathcal{T}^t \otimes \mathbb{1}_{\mathcal{H}}$ in the Koopman operator in $\mathcal{K}$. From the uniqueness of solutions of (6) we can conclude that

$$\mathcal{T}^s U(t, t_0; \theta) = U(t, t_0; \varphi^s(\theta))\, \mathcal{T}^s$$

for all $t, t_0, s \in \mathbb{R}$ and all $\theta \in \mathcal{M}$.

Definition 3. *We define a unitary operator $U_K(t, t_0)$ acting on the enlarged space $\mathcal{K}$ by*

$$U_K(t, t_0) = \mathcal{T}^{-t}\, U(t, t_0; \theta)\, \mathcal{T}^{t_0} = \mathcal{T}^{-(t-t_0)}\, U(t - t_0, 0; \theta).$$

One can show that it is strongly continuous in $t - t_0$, and Stone's theorem implies that there is a self-adjoint operator K on $\mathcal{K}$, called *generalized Floquet Hamiltonian*, such that

$$U_K(t, t_0) = e^{-i\,(t-t_0)K}.$$

The solution of the associated Schrödinger equation

$$i\frac{d}{dt}\psi(t) = K\psi(t) \tag{7}$$

reads $\psi(t) = U_K(t, t_0)\, \psi(t_0) \in \mathcal{K}$, and it is linked to a solution ϕ of the Schrödinger Eq. (6) in $\mathcal{H}$ by

$$\phi(t) = \mathcal{T}^t\, \psi(t) = \psi(t, \varphi^t(\theta)).$$

Proposition 1. *We denote $H(\theta)$ the Hermitian operator on $\mathcal{K}$ which maps $\psi \in \mathcal{K}$ to the function $\theta \mapsto H(\theta)\, \psi(\theta) \in \mathcal{H}$ of $\mathcal{K}$. We assume that $H(\theta)$ is a self-adjoint operator of $\mathcal{K}$. We have the formal equality*

$$K = G + H(\theta).$$

Proof: The operator $U_K(t, t_0)$ is strongly differentiable on $\mathcal{D}(K)$, and we can write formally

$$i\frac{\partial}{\partial t}U_K(t, t_0) = KU_K(t, t_0) \quad \text{for all} \quad t, t_0 \in \mathbb{R}.$$

Therefore

$$\begin{aligned} K &= i\frac{\partial}{\partial t}U_K(t, t_0)|_{t=t_0} \\ &= i\frac{\partial}{\partial t}\big(\mathcal{T}^{-(t-t_0)}\, U(t - t_0, 0; \theta)\big)\big|_{t=t_0} \\ &= i\frac{\partial}{\partial t}\big(\mathcal{T}^{-(t-t_0)}\big)\big|_{t=t_0}\, U(0, 0; \theta) + i\,\frac{\partial}{\partial t}U(t, t_0; \theta)|_{t=t_0} \\ &= G + H(\theta). \end{aligned}$$

□

4. QUANTUM SKEW-PRODUCT ANOSOV PROPERTIES

For a quantum skew-product system defined by the Schrödinger Eq. (6) with a Hamiltonian of the form $H(\hat{x}, \hat{p}, \varphi^t(\theta))$, we define the Anosov property by

Definition 4. *A quantum skew-product system satisfies the quantum skew-product Anosov relations if there exist* $2n$ *continuous functions* $\underline{\alpha}_1, \ldots, \underline{\alpha}_{2n}\colon \mathcal{M} \to V$ *such that the corresponding derivations satisfy for all* $t, t_0 \in \mathbb{R}$ *and* $\theta \in \mathcal{M}$

$$U(t, t_0; \theta)\, L_{\underline{\alpha}_j(\varphi^{t_0}(\theta))}\, U^\dagger(t, t_0; \theta) = e^{\lambda_j(t-t_0)}\, L_{\underline{\alpha}_j(\varphi^t(\theta))}, \tag{8}$$

where $L_{\underline{\alpha}_j(\theta)} = \alpha_{jx}(\theta)^\mathsf{T}\,\hat{x} + \alpha_{jp}(\theta)^\mathsf{T}\,\hat{p}$ *and* λ_j *are* $2n$ *complex numbers such that*

$$\mathrm{Re}(\lambda_1) \le \cdots \le \mathrm{Re}(\lambda_n) < 0 < \mathrm{Re}(\lambda_{n+1}) \le \cdots \le \mathrm{Re}(\lambda_{2n}).$$

The operators $W(\underline{\alpha}(\theta)) = \exp[i(\alpha_x(\theta)^\mathsf{T}\hat{x} + \alpha_p(\theta)^\mathsf{T}\hat{p})]$ and $L_{\underline{\alpha}_j(\theta)}$ define operators $W(\underline{\alpha})$ and $L_{\underline{\alpha}}$ acting on the enlarged Hilbert space $\mathcal{K} = L^2(\mathcal{M}, \mu) \otimes \mathcal{H}$, given by

$$(W(\underline{\alpha})\,\psi)(\theta) = W(\underline{\alpha}(\theta))\,\psi(\theta) \quad \text{for all} \quad \psi \in \mathcal{K},$$

and

$$(L_{\underline{\alpha}_j}\,\psi)(\theta) = L_{\underline{\alpha}_j(\theta)}\,\psi(\theta) \quad \text{for all} \quad \psi \in \mathcal{D}\big(L_{\underline{\alpha}_j}\big) \subset \mathcal{K}.$$

The algebra of observables considered here is the algebra of finite linear combinations of $W(\underline{\alpha})$.

Proposition 2. *The quantum skew-product Anosov relations (8) are satisfied in the Hilbert space $\mathcal{H}$ if and only if in the enlarged space $\mathcal{K} = \mathbb{L}^2(\mathcal{M}, \mu) \otimes \mathcal{H}$, the dynamics generated by the Hamiltonian $K = G + H(\theta)$ satisfies the standard quantum Anosov properties*[1]*:*

$$U_K(t, t_0)\, L_{\underline{\alpha}_j}\, U_K^{\dagger}(t, t_0) \;=\; e^{\lambda_j (t-t_0)}\, L_{\underline{\alpha}_j}.$$

Proof: The quantum skew-product Anosov relations

$$U(t, t_0; \theta)\, L_{\underline{\alpha}_j(\varphi^{t_0}(\theta))}\, U^{\dagger}(t, t_0; \theta) = e^{\lambda_j (t-t_0)}\, L_{\underline{\alpha}_j(\varphi^{t}(\theta))},$$

can be extended as an equality between operator acting on $\mathcal{K}$, by considering the operators as multiplication operators with respect to the variable θ.

Using the equality

$$\mathcal{T}^t L_{\underline{\alpha}_j(\theta)} \mathcal{T}^{-t} = \mathcal{T}^t \alpha_{jx}(\theta)^{\mathsf{T}} \mathcal{T}^{-t} \otimes \hat{x} + \mathcal{T}^t \alpha_{jp}(\theta)^{\mathsf{T}}\, \mathcal{T}^{-t} \otimes \hat{p} = L_{\underline{\alpha}_j(\varphi^t(\theta))}$$

for all $t \in \mathbb{R}$, the quantum skew-product Anosov relations become

$$U(t, t_0; \theta) \mathcal{T}^{t_0} L_{\underline{\alpha}_j} \mathcal{T}^{-t_0} U(t, t_0; \theta) = \mathcal{T}^{-t} e^{\lambda_j (t-t_0)} L_{\underline{\alpha}_j} \mathcal{T}^{t}.$$

According to the Definition 3, we obtain

$$U_K(t, t_0) L_{\underline{\alpha}_j}\, U_K^{\dagger}(t, t_0) = e^{\lambda_j (t-t_0)} L_{\underline{\alpha}_j}.$$

□

5. EXAMPLE OF ALMOST-PERIODIC QUANTUM PARAMETRIC OSCILLATOR

In Ref. 6, it was shown that the upper quantum Lyapunov exponent of the periodic quantum parametric oscillator is positive in the classical instability region. We extend here this result for a wide class of driven quantum parametric oscillators. Moreover we show that, under some conditions, the system verifies the quantum skew-product Anosov properties.

The parametric quantum oscillator is described by the Hamiltonian (we take the mass = 1):

$$H(t) \;=\; \frac{1}{2}\, \hat{p}^2 \;+\; \frac{1}{2}\, f(t)\, \hat{x}^2 \tag{9}$$

where f is an almost-periodic real valued function.

The classical dynamics corresponding to the Hamiltonian (9) has the same form as the eigenvalue equation of the almost-periodic Schrödinger operator:

$$-\ddot{x} \;+\; V(t)\, x = E x \tag{10}$$

with $f(t) = E - V(t)$. The Schrödinger operator $S_{cl} = -d^2/dt^2 + V(t)$ with an almost-periodic potential V is in Weyl's 'limit point' case, and thus admits an essentially unique self-adjoint extension.

We will now analyze the one-parameter family of systems defined by varying E on $\mathbb{C}$ and, in particular, when E is real and in the resolvent set ρ of S_{cl}.

Theorem 1. *For any observable $M \in \mathcal{W}$, in the instability region $E \in \rho \cap \mathbb{R}$, there is a stable direction $\underline{\alpha}_s$, which depends on t_0, for which*

$$\bar{\lambda}_{\underline{\alpha}_s}\left(U, L_{\underline{\alpha}_s}, M, t_0\right) = -\lambda_c < 0,$$

whereas for all other directions $\underline{\alpha}$

$$\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, M, t_0) = \lambda_c > 0.$$

where λ_c is the Lyapunov exponent of the classical system. Thus the upper quantum Lyapunov exponent is positive,

$$\bar{\lambda} = \sup_{\underline{\alpha}} \bar{\lambda}_{\underline{\alpha}} = \lambda_c > 0.$$

Proof: The spectral parameter E is in the resolvent set ρ of the operator S_{cl} if and only if the classical system

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & V(t) - E \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{11}$$

has an exponential dichotomy.[8] In particular, if $E \in \rho$, the system (10) has two linearly independent solutions $q_+ \in \mathbb{L}^2([0, +\infty[)$ and $q_- \in \mathbb{L}^2(] - \infty, 0])$.

The functions

$$m_\pm = \frac{p_\pm}{q_\pm} \quad \text{and} \quad \tilde{m}_\pm = \frac{p_\pm}{q_\pm + i p_\pm}, \quad \text{with} \quad p_\pm = \frac{dq_\pm}{dt},$$

defined for $E \notin \mathbb{R}$ and $E \in \rho \cap \mathbb{R}$, respectively, are almost-periodic.[10,23]
The classical Lyapunov exponent associated with the dynamics of (10) is defined as

$$\lambda_c = \sup\left(\limsup_{t\to+\infty} \frac{1}{2t}\ln(|p|^2 + |q|^2)\right),$$

where the supremum is taken over all non trivial solutions (p, q) of (11), and it satisfies[7]

$$\begin{aligned}\lambda_c &= -\limsup_{t\to+\infty} \frac{1}{2t}\ln(|p_+|^2 + |q_+|^2) \\ &= \limsup_{t\to+\infty} \frac{1}{2t}\ln(|p_-|^2 + |q_-|^2).\end{aligned} \tag{12}$$

In order to determine the upper quantum Lyapunov exponent, we first need to calculate $L_{\underline{\alpha}}(t_0, t)$ which we write in the form

$$L_{\underline{\alpha}}(t, t_0) = \alpha_x(t, t_0)\,\hat{x} \,+\, \alpha_p(t, t_0)\,\hat{p}. \tag{13}$$

The propagator $F(t, t_0)$ of the classical Eq. (11), defined by

$$\begin{pmatrix} p(t) \\ x(t) \end{pmatrix} = F(t, t_0)\begin{pmatrix} p(t_0) \\ x(t_0) \end{pmatrix}, \qquad F(t, t) = 1 \quad \forall t,$$

may be written as

$$F(t, t_0) = P(t)\begin{pmatrix} \frac{\psi_+(t)}{\psi_+(t_0)} & 0 \\ 0 & \frac{\psi_-(t)}{\psi_-(t_0)} \end{pmatrix} P(t_0)^{-1} \tag{14}$$

where $\psi_\pm(t) = q_\pm(t) + ip_\pm(t)$ and

$$P(t) = \begin{pmatrix} \tilde{m}_+(t) & \tilde{m}_-(t) \\ 1 - i\,\tilde{m}_+(t) & 1 - i\,\tilde{m}_-(t) \end{pmatrix}.$$

Using the fact that for quadratic Hamiltonians the Heisenberg equations of motion for the operators $\hat{x}(t)$ and $\hat{p}(t)$ have the same form as the classical equations for $x(t)$ and $p(t)$,

$$\begin{cases} \dfrac{d}{dt}\,\hat{p}(t) = i\,U^\dagger(t, t_0)\,[H, \hat{p}]\,U(t, t_0) \,=\, (V(t) - E)\,\hat{x}(t) \\ \dfrac{d}{dt}\,\hat{x}(t) = i\,U^\dagger(t, t_0)\,[H, \hat{x}]\,U(t, t_0) \,=\, \hat{p}(t), \end{cases}$$

we can write

$$\begin{pmatrix} U^\dagger(t, t_0)\,\hat{p}\,U(t, t_0) \\ U^\dagger(t, t_0)\,\hat{x}\,U(t, t_0) \end{pmatrix} = F(t, t_0)\begin{pmatrix} \hat{p} \\ \hat{x} \end{pmatrix}.$$

Thus, using the relation

$$L_{\underline{\alpha}}(t, t_0) = \begin{pmatrix} \alpha_p \\ \alpha_x \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} U^\dagger(t, t_0)\,\hat{p}\,U(t, t_0) \\ U^\dagger(t, t_0)\,\hat{x}\,U(t, t_0) \end{pmatrix} = \begin{pmatrix} \alpha_p(t, t_0) \\ \alpha_x(t, t_0) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \hat{p} \\ \hat{x} \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \alpha_p(t, t_0) \\ \alpha_x(t, t_0) \end{pmatrix} = (P(t_0)^{-1})^{\mathsf{T}} \begin{pmatrix} \frac{\psi_+(t)}{\psi_+(t_0)} & 0 \\ 0 & \frac{\psi_-(t)}{\psi_-(t_0)} \end{pmatrix} P(t)^{\mathsf{T}} \begin{pmatrix} \alpha_p \\ \alpha_x \end{pmatrix}. \tag{15}$$

From Lemma 0, we can choose $M = W(\underline{\beta}) = e^{i(\beta_x \hat{x} + \beta_p \hat{p})}$ without loss of generality. Then, according to (3),

$$[L_{\underline{\alpha}}(t_0, t), M] = (\alpha_p(t_0, t)\beta_x - \alpha_x(t_0, t)\beta_p)M = -\sigma(\underline{\alpha}(t_0, t), \underline{\beta})M,$$

implying that

$$\|[L_{\underline{\alpha}}(t_0,t),M]\| = |\alpha_p(t_0,t)\beta_x - \alpha_x(t_0,t)\beta_p| = |\sigma(\underline{\alpha}(t_0,t),\underline{\beta})|,$$

where we have used $\| M \| = 1$. By (15), the stable direction $\underline{\alpha}_s$ is given by

$$\begin{pmatrix}\alpha_{ps}\\ \alpha_{xs}\end{pmatrix} = \begin{pmatrix}-q_+(t_0)\\ p_+(t_0)\end{pmatrix} \in \mathbb{R}^2.$$

Indeed we obtain

$$\begin{pmatrix}\alpha_{ps}(t_0,t)\\ \alpha_{xs}(t_0,t)\end{pmatrix} = \psi_+(t)\begin{pmatrix}-1 + i\,\tilde{m}_+(t)\\ \tilde{m}_+(t)\end{pmatrix},$$

and

$$\left\|\left[L_{\underline{\alpha}_s}(t_0,t), A\right]\right\| = |(1 + i\,\tilde{m}_+(t))\,\beta_x \,+\, \tilde{m}_+(t)\,\beta_p|\,|\psi_+(t)|.$$

According to (12), the quantum Lyapunov exponent in this direction is

$$\begin{aligned}\lambda_{\underline{\alpha}_s}\big(U, L_{\underline{\alpha}_s}, A, t_0\big) &= \limsup_{t\to+\infty} \frac{1}{t}\ln(|\psi_+(t)|)\\ &= \limsup_{t\to+\infty} \frac{1}{2t}\ln(|p_+(t)|^2 + |q_+(t)|^2)\\ &= -\lambda_c \,<\, 0.\end{aligned}$$

For all other directions $\underline{\alpha} \in \mathbb{R}^2$, it is easy to check that the upper Lyapunov exponent is positive,

$$\begin{aligned}\lambda_{\underline{\alpha}}(U, L_{\underline{\alpha}}, A, t_0) &= -\limsup_{t\to+\infty} \frac{1}{t}\ln(|\psi_+(t)|)\\ &= -\limsup_{t\to+\infty} \frac{1}{2t}\ln(|p_+(t)|^2 + |q_+(t)|^2)\\ &= \lambda_c > 0.\end{aligned}$$

□

Remark 4. *The result of Theorem 1 can be extended to the multidimensional case where*

$$H(t) \,=\, \frac{1}{2}\,\hat{p}^2 \,+\, \frac{1}{2}\,\hat{x}^{\mathsf{T}} A(t)\,\hat{x}$$

with $A(t)$ a real symmetric matrix depending almost-periodically on time. Writing $A(t) = E\,\mathbb{1} + V(t)$, the equations of motion of the corresponding classical system have the same form as the eigenvalue equation of the Schrödinger operator $S_{cl} = -d^2/dt^2 + V(t)$. In the instability region $E \in \rho \cap \mathbb{R}$, there will be n stable directions $\underline{\alpha}_{s_j}$, depending on t_0, with negative Lyapunov exponent, while *they will be positive for the remaining directions. The main argument is again the exponential dichotomy in the resolvent set.*[9,11]

To study of the Anosov properties for the almost-periodic quantum parametric oscillator, we formulate it as a quantum skew-product system,

$$H(\varphi^t(\theta)) = \frac{1}{2}\,\hat{p}^2 + \frac{1}{2}\,\tilde{f}(\varphi^t(\theta))\,\hat{x}^2 \tag{16}$$

where $\tilde{f}$ is the extension of the almost-periodic function f to a continuous function on its hull and φ^t is the associated minimal flow (see Ref. 10). As before we introduce a parameter E by writing $\tilde{f}(\varphi^t(\theta)) = E - V(\varphi^t(\theta))$, and we denote the hull of the almost-periodic function by $\mathcal{M}$.

The corresponding classical system is now given by

$$\frac{d}{dt}\begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} 0 & E - V(\varphi^t(\theta)) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ x \end{pmatrix}. \tag{17}$$

Definition 5. *A linear system of differential equations*

$$y'(t) = M(\varphi^t(\theta))\,y(t)$$

with $y(t) \in \mathbb{R}^n$, φ^t is a continuous flow on a compact metric space $\mathcal{M}$, and $M(\theta)$ a matrix depending continuously on $\theta \in \mathcal{M}$, is called reducible if it can be transformed into a system with constant coefficients

$$z' = C\,z$$

by a transformation $y(t) = T(\varphi^t(\theta))z(t)$ where $T(\theta)$ is a non singular matrix continuous on $\mathcal{M}$.

Remark 5. *The system (17) is reducible, for instance, when the potential is quasi-periodic with frequencies satisfying a Diophantine condition ([19], Theorem 1.2).*

Theorem 2. *Let the classical system* (17) *be reducible. Then the corresponding quantum parametric oscillator satisfies the quantum skew-product Anosov properties for E being in the resolvent set, $E \in \rho$: there exist two measurable functions $\underline{\alpha}_\pm : \mathcal{M} \to \mathbb{R}^2$ and $\lambda_\pm$ such that* $\pm\,\mathrm{Re}(\lambda_\pm) > 0$ *and*

$$U(t, t_0; \theta)\, L_{\underline{\alpha}_\pm(\varphi^{t_0}(\theta))}\, U^\dagger(t, t_0; \theta) = e^{\lambda_\pm (t - t_0)}\, L_{\underline{\alpha}_\pm(\varphi^t(\theta))},$$

with $L_{\underline{\alpha}_\pm(\theta)} = \alpha_{x\pm}(\theta)\,\hat{x} + \alpha_{p\pm}(\theta)\,\hat{p}$.

Proof: Using reducibility and the hyperbolic character of the flow of (17) in the resolvent set, we obtain

$$F(t, t_0; \theta) = g(\varphi^t(\theta)) \exp\left[(t - t_0)\begin{pmatrix} \lambda_+ & 0 \\ 0 & -\lambda_+ \end{pmatrix}\right] g(\varphi^{t_0}(\theta))^{-1}$$

where g is a is a non singular matrix for all $\theta \in \mathcal{M}$ and $\mathrm{Re}(\lambda_+) \geq 0$. Consequently,

$$\begin{aligned}\begin{pmatrix} U^\dagger(t, t_0; \theta)\, \hat{p}\, U(t, t_0; \theta) \\ U^\dagger(t, t_0; \theta)\, \hat{x}\, U(t, t_0; \theta) \end{pmatrix} &= F(t, t_0; \theta)\begin{pmatrix} \hat{p} \\ \hat{x} \end{pmatrix} \\ &= g(\varphi^t(\theta))\begin{pmatrix} e^{(t-t_0)\lambda_+} & 0 \\ 0 & e^{-(t-t_0)\lambda_+} \end{pmatrix} g(\varphi^{t_0}(\theta))^{-1}\begin{pmatrix} \hat{p} \\ \hat{x} \end{pmatrix}.\end{aligned}$$

Swapping t with t_0 in this equation and using the identity $U^\dagger(t, t_0; \theta) = U(t_0, t; \theta)$, we obtain

$$\begin{aligned} U(t, t_0; \theta)\, L_{\underline{\alpha}(\varphi^{t_0}(\theta))}\, U^\dagger(t, t_0; \theta) &= \begin{pmatrix} \alpha_p(\varphi^{t_0}(\theta)) \\ \alpha_x(\varphi^{t_0}(\theta)) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} U(t, t_0; \theta)\, \hat{p}\, U^\dagger(t, t_0; \theta) \\ U(t, t_0; \theta)\, \hat{x}\, U^\dagger(t, t_0; \theta) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_p(\varphi^{t_0}(\theta)) \\ \alpha_x(\varphi^{t_0}(\theta)) \end{pmatrix}^{\mathsf{T}} g(\varphi^{t_0}(\theta))\begin{pmatrix} e^{-(t-t_0)\lambda_+} & 0 \\ 0 & e^{(t-t_0)\lambda_+} \end{pmatrix} \\ &\quad \times g(\varphi^t(\theta))^{-1}\begin{pmatrix} \hat{p} \\ \hat{x} \end{pmatrix}.\end{aligned}$$

Thus, we can deduce the stable and unstable directions

$$\begin{pmatrix} \alpha_{p-}(\theta) \\ \alpha_{x-}(\theta) \end{pmatrix} = (g(\theta)^{-1})^{\mathsf{T}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha_{p+}(\theta) \\ \alpha_{x+}(\theta) \end{pmatrix} = (g(\theta)^{-1})^{\mathsf{T}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Writing $g = (g_{ij})_{1 \leq i,j \leq 2}$, we obtain

$$\begin{cases} \alpha_{p-}(\theta) = g_{22}(\theta)\, \det(g(\theta))^{-1} \\ \alpha_{x-}(\theta) = -g_{12}(\theta)\, \det(g(\theta))^{-1} \end{cases} \quad \text{and} \quad \begin{cases} \alpha_{p+}(\theta) = -g_{21}(\theta)\, \det(g(\theta))^{-1} \\ \alpha_{x+}(\theta) = g_{11}(\theta)\, \det(g(\theta))^{-1} \end{cases},$$

with

$$U(t, t_0; \theta)\, L_{\underline{\alpha}_\pm(\varphi^{t_0}(\theta))}\, U^\dagger(t, t_0; \theta) = e^{\pm\lambda_+(t-t_0)}\, L_{\underline{\alpha}_\pm(\varphi^t(\theta))}.$$

□

6. QUANTUM ANOSOV PROPERTIES ON A TORUS: THE CONFIGURATIONAL QUANTUM CAT SYSTEM

6.1. Quantum Anosov Properties on a Torus

In this section we study the Anosov properties for systems whose configuration space is a torus, adapting the definitions of Sec. 2 appropriately. The coordinate operators $\tilde{x}_j$ are the self-adjoint operators of multiplication by x_j defined everywhere on the Hilbert space of square integrable functions on the torus of dimension n that we represent by $\mathcal{H} = \mathbb{L}^2([0, 1]^n)$. The momentum operators $\tilde{p}_j = -i \frac{d}{dx_j}$ are unbounded self-adjoint operator defined on the subspaces

$$\mathcal{D}_{p_j} = \{\psi \in \mathcal{H} \mid \psi' \in \mathcal{H} \text{ and } \psi(\ldots, 0, \ldots) = \psi(\ldots, 1, \ldots)\} \subset \mathcal{H}$$

of absolutely continuous (with respect to the variable x_j) functions with periodic boundary conditions.

The coordinate and position operators verify the Heisenberg commutation relations $[x_j, p_k] = \delta_{jk}$ but only on the subspaces

$$\mathcal{D}'_{p_k} = \{\psi \in \mathcal{H} \mid \psi' \in \mathcal{H} \text{ and } \psi(\ldots, 0, \ldots) = \psi(\ldots, 1, \ldots) = 0\}$$

which are not dense in $\mathcal{H}$.

Instead one can work with exponentials of these operators, which can be extended to unitary operators defined on $\mathcal{H}$. Indeed the operators $e^{i\,(\beta^{\mathsf{T}}\tilde{x} + \gamma^{\mathsf{T}}\tilde{p})}$ (defined on an intersection of $\mathcal{D}'_{p_j}$) can be extended as unitary operators for all $\beta \in 2\pi\mathbb{Z}^n$ and all $\gamma \in \mathbb{R}^n$, defined by their action on $\psi \in \mathcal{H}$ as

$$e^{i\,(\beta^{\mathsf{T}}\tilde{x} + \gamma^{\mathsf{T}}\tilde{p})}\, \psi(x) = e^{i\,\beta^{\mathsf{T}}(x+\gamma/2)}\, \psi(x + \gamma). \tag{18}$$

We choose as algebra of observables the algebra $\mathcal{A}$ generated by the operators $e^{i\,(\beta^{\mathsf{T}}\tilde{x} + \gamma^{\mathsf{T}}\tilde{p})}$ for all $\beta \in 2\pi\mathbb{Z}^n$ and all $\gamma \in \mathbb{R}^n$. These operators satisfy the Weyl form of the commutation relations (1,2), therefore $\mathcal{A}$ is isomorphic to the subalgebra $\mathcal{W}_T$ of $\mathcal{W}$ defined by restricting the coefficients of the Weyl operators $W(\beta, \gamma)$ such that $\beta \in 2\pi\mathbb{Z}^n$. We denote this isomorphism by $\phi : \mathcal{A} \to \mathcal{W}_T \subset \mathcal{W}$.
We assume that the dynamics defines an automorphism of $\mathcal{A}$:

$$\tau(t, t_0) : M \mapsto U(t, t_0) M U^{\dagger}(t, t_0) \quad \forall M \in \mathcal{A}, \forall t, t_0 \in \mathbb{R},$$

where $U(t, t_0)$ denotes the unitary propagator on $\mathcal{H}$ with initial time t_0.

As we recall in the Remark 3., the Anosov properties give a representation of the Anosov group[24] as a group of endomorphisms on the algebra of observables $\mathcal{A}$. Here we consider the translation automorphisms for all $\underline{\alpha} = (\alpha_x, \alpha_p) \in \mathbb{R}^{2n}$ defined by their action on the operators $e^{i\,(\beta^{\mathsf{T}}\tilde{x} + \gamma^{\mathsf{T}}\tilde{p})} \in \mathcal{A}$, by

$$\sigma_{\underline{\alpha}}(s) \,:\, e^{i\,(\beta^{\mathsf{T}}\tilde{x}+\gamma^{\mathsf{T}}\tilde{p})} \mapsto\; e^{is\left(\alpha_x^{\mathsf{T}}\gamma - \alpha_p^{\mathsf{T}}\beta\right)} e^{i(\beta^{\mathsf{T}}\tilde{x}+\gamma^{\mathsf{T}}\tilde{p})}.$$

Definition 6. *A system on a torus satisfies the quantum Anosov relations if there are* $2n$ *directions* $\underline{\alpha}_1, \ldots, \underline{\alpha}_{2n} \in \mathbb{R}^{2n}$ *such that for all* $t, t_0 \in \mathbb{R}$

$$\tau(t, t_0)\, \sigma_{\underline{\alpha}_j}(s) = \sigma_{\underline{\alpha}_j}\left(s e^{\lambda_j (t-t_0)}\right)\, \tau(t, t_0). \tag{19}$$

where λ_i are $2n$ complex numbers such that

$$\mathrm{Re}(\lambda_1) \leq \cdots \leq \mathrm{Re}(\lambda_n) < 0 < \mathrm{Re}(\lambda_{n+1}) \leq \cdots \leq \mathrm{Re}(\lambda_{2n}).$$

These relations are algebraic, therefore we can express them in the algebra $\mathcal{W}$ using the isomorphism between $\mathcal{A}$ and $\mathcal{W}_T$. We denote $\tau_w(t, t_0)$ the automorphisms of $\mathcal{W}_T$ defined by $\tau_w(t, t_0) = \phi\tau(t, t_0)\,\phi^{-1}$. We assume that $\tau_w(t, t_0)$ can be extended to an automorphism of $\mathcal{W}$ (i.e. for β real and not only in $\beta \in 2\pi\mathbb{Z}^n$).

The derivations can be expressed in terms of the translation operators of $\mathcal{W}$ as follows: for all $W(\underline{\beta}) \in \mathcal{W}_T$, $\underline{\beta} = (\beta, \gamma)$,

$$\sigma_{\underline{\alpha}}(s)(W(\underline{\beta})) = e^{isL_{\underline{\alpha}}}\, W(\underline{\beta})\, e^{-isL_{\underline{\alpha}}} = e^{is\left(\alpha_x^{\mathsf{T}}\gamma - \alpha_p^{\mathsf{T}}\beta\right)}\, W(\underline{\beta}) \in \mathcal{W}_T$$

where we have used (2), and the notation $W(\underline{\alpha}) = e^{i\,L_{\underline{\alpha}}}$.

We remark that the derivations defined here are not inner derivations, i.e. we do not impose $\alpha_x \in 2\pi\mathbb{Z}^2$. According to (3), the derivations are well defined for all $M \in \mathcal{W}_T$ by $\delta_{\underline{\alpha}}(M) = [L_{\underline{\alpha}}, M] \in \mathcal{W}_T$ for all $\underline{\alpha} = (\alpha_x, \alpha_p) \in \mathbb{R}^{2n}$.

These remarks prove the following

Lemma 2. *A system on a torus (such that $\tau_w(t, t_0)$ can be extended to an automorphism of $\mathcal{W}$) satisfies the quantum Anosov relations, if there are $2n$ directions $\underline{\alpha}_1, \ldots, \underline{\alpha}_{2n} \in \mathbb{R}^{2n}$ such that for all $t, t_0 \in \mathbb{R}$*

$$\tau_w(t, t_0)(e^{i\,s\,L_{\underline{\alpha}_j}}) = e^{i\,s\,e^{\lambda_j(t-t_0)}\,L_{\underline{\alpha}_j}}, \tag{20}$$

where λ_i are $2n$ complex numbers such that

$$\mathrm{Re}(\lambda_1) \leq \cdots \leq \mathrm{Re}(\lambda_n) < 0 < \mathrm{Re}(\lambda_{n+1}) \leq \cdots \leq \mathrm{Re}(\lambda_{2n}).$$

Remark 6. *These definitions can also be formulated for discrete time unitary dynamics:*

$$\tau(k) : M \mapsto U_F^k\, M\, (U_F^\dagger)^k \in \mathcal{A} \quad \forall\, M \in \mathcal{A},\ \forall\, k \in \mathbb{Z},$$

In this case, the Anosov conditions (19) and (20) become respectively

$$\tau(k)\,\sigma_{\underline{\alpha}_j}(s) = \sigma_{\underline{\alpha}_j}(s\,e^{k\,\lambda_j})\,\tau(t, t_0) \quad \textit{and}$$

$$\tau_w(k)(e^{i\,s\,L_{\underline{\alpha}_j}}) = e^{i\,s\,e^{k\,\lambda_j}\,L_{\underline{\alpha}_j}}.$$

6.2. Example of Anosov System on a Torus: The Configurational Quantum Cat System

We consider a charged particle of mass $m = 1$ constrained to move in a unit square with periodic boundary conditions (period 1) subject to external periodic time dependent electromagnetic fields. It was shown in Refs. 25, 26 that the external fields can be chosen in such a way that the configuration space of the particle is mapped periodically to itself according to Arnold's cat map. This system has a discrete time dynamics given by the time evolution operator over one period T or Floquet operator $U_F = U(T, 0)$. The time evolution is thus defined by the iteration of the operator U_F acting on the Hilbert space $\mathcal{H} = \mathbb{L}^2([0, 1]^2)$:

$$U_F = e^{-\frac{iT}{2}\tilde{p}^2}\, e^{-\frac{i}{2}(\tilde{x}^\top V^\top \tilde{p} + \tilde{p}^\top V \tilde{x})}$$

with $\tilde{p} = \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}$, $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$, and V is a matrix such that $\exp(V)$ is Arnold's cat map:

$$e^V = C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The Floquet operator U_F is the product of two unitary operators: The first one, $e^{-\frac{iT}{2}\tilde{p}^2}$, describes the free-particle propagation during a time interval T. The second factor, $D_V = e^{-\frac{i}{2}(\tilde{x}^\top V^\top \tilde{p} + \tilde{p}^\top V \tilde{x})}$, is the kick operator which acts during an infinitesimally short-time interval. D_V can be extended to unitary operator of dilatation defined on $\mathcal{H}$ by:

$$D_V\, \psi(x) = e^{-\operatorname{tr}(V)/2}\, \psi(e^{-V} x) = e^{-\frac{3}{2}}\, \psi(C^{-1}x). \tag{21}$$

Lemma 3. *The quantum dynamics given by U_F is a well defined automorphism $\tau(k)$ on the algebra of observables $\mathcal{A}$, in particular U_F satisfies for all $M \in \mathcal{A}$*

$$\tau(k)(M) = U_F^k\, M\, (U_F^\dagger)^k \in \mathcal{A}.$$

Moreover the automorphism $\tau_w(k)$ of $\mathcal{W}_T$, defined by isomorphism from $\tau(k)$, can be extended to an automorphism of $\mathcal{W}$.

Proof: According to (21) and (18), the operator D_V satisfies

$$D_V^\dagger\, e^{i(\beta^\top \tilde{x} + \gamma^\top \tilde{p})}\, D_V = e^{i(\beta^\top C^{-1}\tilde{x} + \gamma^\top C\tilde{p})}.$$

The evolution over one period T is thus given for all $\beta \in 2\pi\mathbb{Z}^2$ and all $\gamma \in \mathbb{R}^2$ by

$$\begin{aligned}\tau(1)(e^{i\left(\beta^\top \tilde{x} + \gamma^\top \tilde{p}\right)}) &= U_F e^{i\left(\beta^\top \tilde{x} + \gamma^\top \tilde{p}\right)} U_F^\dagger \\ &= e^{i(\beta^\top C^{-1}\tilde{x} + (-T\beta^\top C^{-1} + \gamma^\top C)\tilde{p})} \in \mathcal{A}\end{aligned}$$

since C is a unimodular matrix and thus $\beta^\mathsf{T} C^{-1} \in 2\pi\mathbb{Z}^2$.

Therefore in $\mathcal{W}_T$ the automorphism is given by

$$\tau_w(1)(W(\beta,\gamma)) = W(C^{-1}\beta,\ -T\,C^{-1}\beta\,+\,C\gamma) \tag{22}$$

which can be extended as a automorphism of $\mathcal{W}$ by replacing $\beta \in 2\pi\mathbb{Z}^2$ by $\beta \in \mathbb{R}^2$. □

Theorem 3. *The configurational quantum cat system satisfies quantum discrete time Anosov properties: There exist two stable directions* $\underline{\alpha}_1$ *and* $\underline{\alpha}_2$*, such that*

$$\tau_w(k)(e^{i\,s\,L_{\underline{\alpha}_j}}) = e^{i\,s\,e^{-k\lambda}\,L_{\underline{\alpha}_j}} \qquad j = 1,2,$$

and two unstable directions $\underline{\alpha}_3$ *and* $\underline{\alpha}_4$*,*

$$\tau_w(k)(e^{i\,s\,L_{\underline{\alpha}_j}}) = e^{i\,s\,e^{k\lambda}\,L_{\underline{\alpha}_j}} \qquad j = 3,4,$$

where $\lambda > 0$ *is such that* $e^{\pm\lambda}$ *are the eigenvalues of Arnold's cat map* C*.*

Proof: In order to avoid complicate notation, we write $W(s\,\underline{\alpha}_j)$ instead of $e^{isL_{\underline{\alpha}_j}}$.

Equation (22) allows one to conclude that

$$\tau_w(1)(e^{i\,s\,L_{(0,\alpha_p)}}) = \tau_w(1)(W(0,s\alpha_p)) = W(0,\ s\,C\,\alpha_p).$$

Therefore $\underline{\alpha}_1 = (0, v_-)$ and $\underline{\alpha}_3 = (0, v_+)$ are respectively stable and unstable directions, where $v_\pm$ are the eigenvectors of C with $Cv_\pm = e^{\pm\lambda}v_\pm$.

The other pair of stable and unstable directions are $\underline{\alpha}_2 = ((C^2 - \mathbb{1})\,v_+,\ T\,v_+)$ and $\underline{\alpha}_4 = ((C^2 - \mathbb{1})\,v_-,\ T\,v_-)$. Indeed, using the Eq. (22), we observe that

$$\begin{aligned}
&\tau_w(1)(W(s(C^2 - \mathbb{1})v_\pm, sTv_\pm))\\
&= W(sC^{-1}(C^2 - \mathbb{1})v_\pm, s(-TC^{-1}(C^2 - \mathbb{1}) + TC)v_\pm)\\
&= W(s(C^2 - \mathbb{1})C^{-1}\,v_\pm, sTC^{-1}v_\pm)\\
&= W(se^{\mp\lambda}(C^2 - \mathbb{1})v_\pm, se^{\mp\lambda}Tv_\pm).
\end{aligned}$$

□

Remark 7. *The derivations* $\delta_{\underline{\alpha}_1}$ *and* $\delta_{\underline{\alpha}_3}$ *are inner derivations, but* $\delta_{\underline{\alpha}_2}$ *and* $\delta_{\underline{\alpha}_4}$ *are not because the coefficients of each eigenvector* $v_\pm$ *are rationally independent.*

Remark 8. *The Lyapunov exponents for systems on the torus can be defined by adapting Definition 1, and using the isomorphism* $\phi : \mathcal{A} \to \mathcal{W}_T \subset \mathcal{W}$

$$\bar{\lambda}_{\underline{\alpha}}(U, L_{\underline{\alpha}}, M, t_0) := \limsup_{t\to\infty} \frac{1}{t}\ \ln \|\,\phi^{-1}([L_{\underline{\alpha}}, \phi(\tau(t,t_0)(M))])\,\|,$$

where $M \in \mathcal{A}$ *and* $\|M\| = \sup_{\psi\in\mathcal{H}} \|M\psi\|/\|\psi\|$.

It follows immediately from the Anosov properties that the upper Lyapunov exponent for the configurational Arnold cat is $\overline{\lambda} = \lambda > 0$.

7. SUMMARY AND DISCUSSION

If one searches for quantum mechanical systems with a genuinely complicated time evolution, non-autonomous systems are promising candidates, in particular for strong dependence on initial conditions. In order to distinguish between potentially different types of quantum dynamics, we have discussed two concepts in this paper. Our definitions of a quantum mechanical Lyapunov exponent and of quantum Anosov relations adapt existing notions in such a way that they apply not only to autonomous but to driven quantum systems as well. By their very construction, these concepts are intrinsically quantum mechanical: no reference to a classical counterpart of the systems is made in the definitions. This is an important feature since the discussion of a quantum system should, in our view, rely as little as possible on classical notions.

The quantum Lyapunov exponent and the quantum Anosov property are indeed appropriate to distinguish qualitatively different quantum dynamics. To see this, we have considered two specific solvable models the dynamics of which is well understood: the quantum parametric oscillator and the configurational quantum cat map. In the case of the driven oscillator, for example, we have found that the quantum Lyapunov exponents are non-zero for those parameter values where the time evolution of the corresponding classical oscillator is unstable. We have shown that it is possible to devise quantum mechanical tools which make rigorous the existence of qualitatively different dynamics for non-autonomous quantum systems.

The two quantum systems used as examples to illustrate these concepts are constructed using classical models. This is motivated by the fact that they allow explicit calculations for the quantum models. We emphasize, however, that the Lyapunov exponents are defined in an intrinsically quantum mechanical way. If one were to add small perturbations in these models, we expect that the non-zero Lyapunov exponents will continue to exist while the exact equivalence to classical systems will be destroyed.

Finally, we should emphasize that our approach is complementary to a substantial amount of work done in the field of quantum chaos where one succeeds to describe important properties of a quantum system such as energy levels and their statistics in the semiclassical limit. In this approach, the link to the classical counterpart of the quantum system under study is a vital ingredient while the phenomena of interest do not necessarily have an interpretation of chaos in the sense of dynamical systems. Our results point in a different direction: they indicate that there are quantum tools which allow one to distinguish qualitatively different quantum dynamics just as there are classical tools to distinguish between classical regular and chaotic dynamical systems.

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